6.5 Graph powers and the transative closure

The Graph power theorem

- Let G be a directed graph. Let u and v be any 2 vertices in G. There is an edge from u to v in G^k i.f.f. there is a walk of length k from u to v in G.

- The relation R^+ is the transative closure of R and is the smallest relation that is both transative and includes all pairs of R.

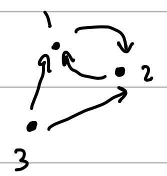
$$G^+ = G^1 \cup G^2 \cup G^3 \cup \ldots \cup G^n$$

- G^+ contains every edge in $G^1, \ldots, G^n$.
- G^+ is the transative closure of G.

- Procedure to find the transative closure of a relation R on a set A.
  - Repeat following step until no pair is added to R.
    - If there are three elements $x, y, z \in A$ such that $(x,y) \in R$, $(y,z) \in R$ and $(x,z) \notin R$, then add (x,z) to R.

6.6 Matrix multiplication and graph powers

- Adjacency matrix



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- The entry is 1 if there is an edge from vertex i to vertex j.

Theorem

- Let G be a directed graph with n vertices and let A be the adjacency matrix for G. Then for any $k \geq 1$, A^k is the adjacency matrix of G^k, where Boolean addition and multiplication are used to compute A^k.

Theorem

- Let G and H be two directed graphs with the same vertex set. Let A be the adjacency matrix for H. Then the adjacency matrix for $G \cup H$ is $A+B$, where Boolean addition is used on the entries of matrices A and B.

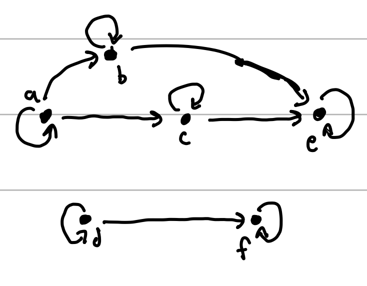
6.7 Partial Orders

- A relation R on a set A is a partial order if its reflexive, transitive, and anti-symmetric.
  - Partial order notation: $a \preceq b$, is used to express aRb
    - read "a is at most b"
- The domain along with a partial order defined on it is denoted $(A, \preceq)$ and is called a partially ordered set or poset.

- Two elements of a partially ordered set, x and y, are comparable if $x \preceq y$ or $y \preceq x$. Otherwise the elements are incomparable.

- A partial order is a total order if every two elements in the domain are comparable.

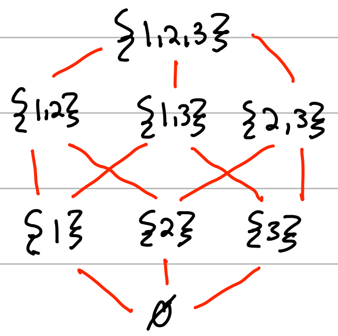
- An element x is a minimal element if there is no $y \neq x$ such that $y \preceq x$.
- An element x is a maximal element if there is no $y \neq x$ such that $x \preceq y$.



minimal elements: a and d because there are no arrows pointing into them except the ones from themselves.

maximal element: e and f because the only edges leaving e and f point to themselves.

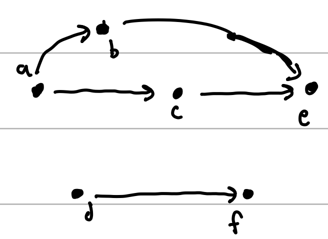
- An upward drawing or Hasse diagram is a useful way to depict a partial order on a finite set.



6.8 Strict orders and directed acyclic graphs

- A relation R is a strict order if R is transative and anti-reflexive.
- Strict order notation $a \prec b$ is used to express aRb
    - Read "a is less than b"
- The domain along with a strict order defined on it is denoted $(A, \prec)$

- Two elements x and y are comparable if $x \prec y$ or $y \prec x$. Otherwise, the elements are incomparable.
- A strict order is a total order if every distinct pair of elements is comparable.
- An element x is minimal if there is no y such that $y \prec x$.
- An element x is maximal if there is no y such that $x \prec y$.



minimal elements: a and d because there are no arrows into a or d.

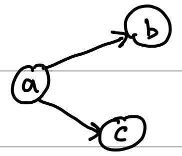
maximal elements: e and f because there are no arrows out of e or f.

- A directed acyclic graph (DAG) is a directed graph that has no cycles.

Theorem

- Let G be a directed graph. G has no cycles i.f.f. G^+ is a strict order.

- A <u>topological</u> sort for a DAG is an ordering of the vertices that is consistent with the edges in the graph. That is, if there is an edge (u,v), then u appears earlier than v in the topological sort.



<u>two topological sorts</u>

a, b, c

a, c, b

6.9 Equivalence relations

- A relation R is an equivalence relation if R is reflexive, symmetric, and transative.
- Notation: $a \sim b$ is used to express aRb
  - Read "a is equivalent to b"

- If A is the domain of an equivalence relation and $a \in A$, then $[a]$ is defined to be the set of all $x \in A$ such that $a \sim x$.
  - The set $[a]$ is called an equivalence class.

Theorem

Consider an equivalence relation on a set A. Let $x, y \in A$.

- If $x \sim y$ then $[x] = [y]$
- If it is not the case that $x \sim y$, then $[x] \cap [y] = \emptyset$

- A set of sets is <u>pairwise disjoint</u> if the intersection of any pair of the sets is empty.